ELEC2221
Digital Systems and Signal Processing
SeungHwan Won (원승환, 元勝煥)
Communications: Random Signals (III)
Learning Outcomes

By the end of this session you should be able to:

• Characterise a signal using auto- and cross-correlation;
• Characterise a stationary and ergodic signal;
Revision

Last session, we discussed the following:

- Cumulative Distribution function
- Moment, mean and variance of a random variable
- Random process: mean and variance

*Any questions?*
Cross-correlation and auto-correlation of deterministic sequences (1)

Let $x[\cdot]$, $y[\cdot]$ be two non-random sequences of “finite energy” ($\sum_{k=-\infty}^{\infty} x[k]^2 < \infty$). In order to measure how much they look alike, we define the cross-correlation sequence of $x[\cdot]$ and $y[\cdot]$: 

$$r_{xy}[\tau] := \sum_{k=-\infty}^{\infty} x[k] y[k - \tau]$$

The index $\tau$ for $r_{xy}[\cdot]$ is called the lag.

When $x[\cdot] = y[\cdot]$, we measure the “similarity” between different shifts of the same sequence. Then

$$r_{xx}[\tau] := \sum_{k=-\infty}^{\infty} x[k] x[k - \tau]$$

is called the auto-correlation sequence of $x[\cdot]$. 
Cross-correlation and auto-correlation of deterministic sequences (2)

For example, consider the finite-length sequence (assumed zero outside of this interval)


We derive from it the sequence \( y[n] = x[n-2] + g[n] \) where \( g[\cdot] \) is a Gaussian sequence with mean zero and variance 1. We generate two \( y[\cdot] \) sequences, and compute their cross-correlation with \( x[\cdot] \).

From \( y[n] = x[n - 2] + g[n] \) we conclude that \( y[\cdot] \) should “look like” \( x[\cdot - 2] \). This is confirmed by the value of the cross-correlation function.
Yes, well, but... how does auto-/cross-correlation works? (1)

\[ r_{xx}[\tau] := \sum_{k=-\infty}^{\infty} x[k]x[k - \tau] \]

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The value of the “inner product” in this case is

\[ 21 - 7 - 3 = 11 \]
Yes, well, but... how does auto-/cross-correlation works? (2)

\[ r_{xx}[\tau] := \sum_{k=-\infty}^{\infty} x[k] x[k - \tau] \]

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The value of the “inner product” in this case is

\[ 33 + 77 - 4 + 12 = 118 \]
Yes, well, but... how does auto-/cross-correlation works? (3)

\[ r_{xx}[\tau] := \sum_{k=-\infty}^{\infty} x[k]x[k - \tau] \]

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The value of the “inner product” in this case is

\[ 9 + 121 + 49 + 1 + 16 + 9 = 205 \]
Yes, well, but... how does auto- / cross- correlation works? (4)

\[ r_{xx}[\tau] := \sum_{k=-\infty}^{\infty} x[k]x[k - \tau] \]

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<td>x[k - 1]</td>
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The value of the “inner product” in this case is

\[ 33 + 77 - 4 + 12 = 118 \]

(Not surprising, since interchanging the two vectors I get exactly what I was dealing with two slides ago...)

The maximum will always be achieved for the case in which \( x[\cdot] \) and its shifted version will be perfectly aligned...
Let \( x[\cdot] \) be a random signal; then the correlation between two samples \( x[n] \) and \( x[m] \) is defined as

\[
r_{xx}[n, m] := \mathcal{E}\{x[n] \cdot x[m]^*\}
\]

This expression defines a function of \( n \) and \( m \), called the **auto-correlation of \( x[\cdot] \)**.

The \( x[m]^* \) means “conjugate of \( x[m] \)”. If \( x[\cdot] \) is real, forget it. However, often engineers use complex signals!
Auto-correlation function (1)

The autocorrelation function of a continuous-time random process $X(t)$ is defined as the expected value of the product $X(t_1)X(t_2)$:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p_{X_1, X_2}(x_1, x_2; t_1, t_2) \, dx_1 dx_2$$

For a discrete-time random process, the autocorrelation function is defined as:

$$R_{XX}[n_1, n_2] = E[X[n_1]X[n_2]] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p_{X_1, X_2}(x_1, x_2; n_1, n_2) \, dx_1 dx_2$$

The autocorrelation function describes the relationship between two samples of a random process.
Auto-correlation function (2)
Auto-correlation function (3)
Auto-correlation function (4)
Do the statistical properties (mean, variance, etc.) of the random signal depend on $n$?

Do the “rules of the game” change over time, or not?

How much of the statistical properties of the ensemble are reflected in those of a realization?

If I make a finite number of experiments, can I draw from them some conclusion about the random process producing the measurements?
Stationarity: the “rules of the game” don't change (1)

If the “statistical description” of a random signal is independent of $n$, then the signal is said to be strictly stationary (SS). Note that the signal on p. 19 is not stationary.

Specifically, for a random process $X[n]$ and a shift $\tau$, $X[n]$ is SS if

$$P_{X_1,X_2,...}(X_1[n_1], X_2[n_2], ...) = P_{X_1,X_2,...}(X_1[n_1 + \tau], X_2[n_2 + \tau], ...)$$

Formally, a random signal is SS if all the moments $E\{x^{k_0}[n] \cdot x^{k_1}[n + l_1] \cdots x^{k_L}[n + l_L]\}$ for any selection of the powers $k_i$, $0 \leq i \leq L$ and any value of $L$, are independent of $n$ (i.e. depend only on the intersample spacing).
Stationarity: the “rules of the game” don't change (2)

$L$ specifies the length of the “comb”; the indices $i_j, j = 1, \ldots, L$, represent the “teeth” which are present in the “comb”, and $k_m, m = 0, \ldots, L$, denotes the “lengths” of the “teeth”. Let us consider $L = 6, i_0 = 0, i_1 = 3, i_2 = 5$, etc.

Stationarity means that choosing any type of “comb”, sliding it along the time axis, and computing the mean and autocorrelation, always gives the same answer.
Stationarity: Questions (2)

**Question**: Consider the random signal $x[n]$ as $x[n] := 2nv[n] - 5$, where $v[n]$ is obtained by sampling independently a uniformly distributed R.V. in the interval $[-1; 4]$. Compute the average of $x[n]$.

**Answer**: First of all, recall that the expectation operator is linear: $\mathbb{E}\{ax[n] + \beta y[n]\} = a\mathbb{E}\{x[n]\} + \beta \mathbb{E}\{y[n]\}$.

The average of $x[n]$ is

$$\mathbb{E}\{x[n]\} = 2n\mathbb{E}\{v[n]\} - 5 = 2n\frac{3}{2} - 5 = 3n - 5$$

since the mean of a uniformly distributed variable in $[a, b]$ is $\frac{a+b}{2}$. Observe that in this case the average of $x[n]$ depends on $n$. 

Excerpted from previous lecture slides
In order to calculate the mean or moment of a random process, it is necessary to perform an ensemble average. In many cases, this may not be possible as we may not be able to observe all or a large number of realisations. Hence, if a stationary random process has the time average of a particular function equal to the ensemble average, then the process is ergodic.

For example, consider a stationary random process \( X(t) \) with a mean \( \mu_X \) and we can observe one realisation \( x(t) \). For an ergodic process, we can obtain \( \mu_X \) as the time average of the realisation \( x(t) \).

A natural consequence of ergodicity is that in measuring various statistical averages such as mean and autocorrelation, it is surely sufficient to look at one realisation of the process and find the corresponding time average, rather than considering a large number of realisations and averaging over them.
Ergodicity is an important property which allows us to use one realization in order to study properties of the whole ensemble (the random signal): throwing 10000 dice to calculate the mean is replaced by throwing 1 die 10000 times.

Observe that if $x[\cdot]$ is ergodic, then it is stationary (the signal average is a constant, and cannot be equal to the ensemble average if the latter is a function of $n$). The converse implication does not hold true.
Question: Consider the random signal \( x[n] = A \sin(\omega n + \phi) \) where \( A \) and \( \phi \) are statistically independent random variables and the frequency \( \omega \) is a constant. \( A \) is uniformly distributed between 0 and 10, while \( \phi \) is uniformly distributed between 0 and \( \pi \). Is this random signal stationary? Ergodic?
**Question:** Consider the random signal $x[n] = A \sin(\omega n + \phi)$ where $A$ and $\phi$ are statistically independent random variables and the frequency $\omega$ is a constant. $A$ is uniformly distributed between 0 and 10, while $\phi$ is uniformly distributed between 0 and $\pi$. Is this random signal stationary? Ergodic?

**Answer:** We first compute $\mathcal{E}\{x[n]\} = \mathcal{E}\{A\} \mathcal{E}\{\sin(\omega n + \phi)\}$ (the equality holds since $A$ and $\phi$ are independent). Now

$$\mathcal{E}\{A\} = \int_0^{10} a f_A(a) da = \int_0^{10} a \cdot \frac{1}{10} da = 5$$

while

$$\mathcal{E}\{\sin(\omega n + \phi)\} = \int_0^{\pi} \sin(\omega n + \phi) \frac{1}{\pi} d\phi = -\frac{1}{\pi} \cos(\omega n + \phi) \bigg|_0^{\pi} = \frac{2}{\pi} \cos(\omega n)$$

Conclude that $\mathcal{E}\{x[n]\} = \frac{10}{\pi} \cos(\omega n)$, which depends on $n$. The signal is not stationary, and hence, not ergodic.
Measurement of Mean and Variance

- To practically calculate mean and variance, we assume ergodicity: ensemble averages can be swapped against time averages.

- Example: throwing 10000 dice to calculate the mean is replaced by throwing 1 die 10000 times, hoping that the result will be the same.

- Time averages instead of expectations (using PDF):

\[
\mu_x = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] ; \quad \sigma_x^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \mu_x)^2
\]

- For continuous time variables, summations are replaced by integration.

- Note that for zero mean signals ($\mu_x = 0$), the variance represents the power of the signal $x[n]$. 
Summary

- Autocorrelation, cross-correlation;
- Stationarity, ergodicity properties.

Any questions ??? 😊
Learning Outcomes

By the end of this session you should be able to:

• Analyse the auto- and cross-correlation of stationary and ergodic signals;
• Understand the application of correlation in real systems;
• Evaluate the power spectral density of a random signal.
Revision

Last session, we discussed the following:

• Autocorrelation, cross-correlation;
• Stationarity, ergodicity properties.

Any questions?
Auto-correlation of stationary signals (1)

Let \( x[\cdot] \) be a random signal; then the auto-correlation between two samples \( x[n] \) and \( x[m] \) is defined as

\[
\begin{align*}
    r_{xx}[n, m] & := \mathcal{E}\{x[n] \cdot x[m]^*\}
\end{align*}
\]

If a signal is stationary, then the joint distribution of \( x[n] \) and \( x[m] \) depends only on the “spacing” between the two samples \( m \) and \( n \) (i.e. \( n - m \)). This implies that \( \mathcal{E}\{x[n] \cdot x[m]^*\} \) also only depends on \( n - m \), and consequently

\[
\begin{align*}
    r_{xx}[n, m] & = r_{xx}[n - m]
\end{align*}
\]

The difference \( \tau := n - m \) is called the lag.
In general, under the assumption of stationarity, the larger the lag, the smaller the value of the autocorrelation function. This agrees with our intuition; take for example two specific specific lags $\tau = 3$ (left) and $\tau = 50$ (right):

See the next slide to investigate these two curves in detail.

Consider that in the next slide the curves on the left look “similar”, the ones on the right “dissimilar”.
Auto-correlation of stationary signals (3)
Wide-sense stationary signals (1)

Since we most often use least-squares methods, we are normally interested in an analysis of the statistics of a process up to the second moments. In this case, we do not need (strict) stationarity, and we settle for

• A constant mean: $\mathcal{E}\{x[n]\} = \mu$ for all $n$;

• A correlation function depending only on the lag $\tau$.

A signal enjoying these two properties is called a wide-sense stationary (WSS) random process.
Wide-sense stationary signals (2)

Since we most often use least-squares methods, we are normally interested in an analysis of the statistics of a process up to the second moments. In this case, we do not need (strict) stationarity, and we settle for

- A constant mean: $\mathcal{E}\{x[n]\} = \mu$ for all $n$;

- A correlation function depending only on the lag $\tau$.

A signal enjoying these two properties is called a wide-sense stationary random process.

If a WSS signal is input to a linear, time-invariant filter, also the output is WSS.

(Why: we have seen what happens to the mean of the output, we will shortly see what happens to the correlation function.)
Parameter estimation

Input signal \( u(t) \) → Physical system → Output signal \( y(t) \)

Parameter estimator for mathematical model

Model parameters (parameter vector) \( \Theta \)
The method of **Least Squares (LS)** is a standard approach to the approximate solution of over-determined systems, i.e., sets of equations in which there are more equations than unknowns. "Least squares" means that the overall solution **minimises the sum of the squares of the errors** made in the results of every single equation.

When the parameters appear linearly in these expressions then the LS estimation problem can be solved in closed form, and it is relatively straightforward to derive the statistical properties for the resulting parameter estimates.

The classical example of a LS estimator is fitting a straight line \( f(x) = p_1 + p_2 x \) to a set of \( N \) measurements \( \{x_i, f_i\} \) for \( i = 1, \ldots, N \).

In case all points lie exactly on the straight line we have: \( f_i = p_1 + p_2 x_i \) for all \( i = 1, \ldots, N \).

http://en.wikipedia.org/wiki/Least_squares
Least-square method (2)

**LS estimation of straight line**

```matlab
x = -5:.1:5; % the x-values
f = 3.2 + 1.4*x; % the f(x) values
f = f + randn(size(f)); % add noise
M = [ length(x) sum(x); sum(x) sum(x.^2) ];
vf = [ sum(f) sum(x.*f) ]';
p = inv(M)*vf;

plot(x, f, 'o', x, p(1)+p(2)*x, 'r-');
```

**LS Estimation:**

The points are generated to lie on the line $f(x) = 3.2 + 1.4x$. Normal distributed noise is added.

The best line fit is given by $f(x) = 3.2357 + 1.3800x$. 

Least Squares Estimation Techniques (An introduction for computer scientists) By Rein van den Boomgaard
Inclusion relationships

Random process

Wide-sense stationary random process

Strictly stationary random process

Ergodic random process
Question: Consider the signal $x[\cdot]$ where $x[n]$ is independently distributed and with mean $\mu$ and variance $\sigma^2$ for all $n$. Is this signal wide-sense stationary?
**Question:** Consider the signal $x[·]$ where $x[n]$ is independently distributed and with mean $\mu$ and variance $\sigma^2$ for all $n$. Is this signal wide-sense stationary?

**Answer:** The correlation between the random variables $x[n]$ and $x[m]$ equals

$$\mathbb{E}\{x[n]x[m]\} = \begin{cases} 
\mu^2 & \text{if } m \neq n \\
\sigma^2 + \mu^2 & \text{if } m = n 
\end{cases}$$

The second equality follows from

$$\mathbb{E}\{x[n]x[n]\} = \mathbb{E}\{(x[n] - \mu + \mu)(x[n] - \mu + \mu)\}$$
$$= \mathbb{E}\{(x[n] - \mu)(x[n] - \mu) + 2\mu(x[n] - \mu) + \mu^2\}$$
$$= \mathbb{E}\{(x[n] - \mu)(x[n] - \mu)\} + 2\mu\mathbb{E}\{(x[n] - \mu)\} + \mu^2$$
$$= \sigma^2 + 2\mu^2 - 2\mu^2 + \mu^2$$

Conclude from this that the correlation of $x[·]$ depends only on the difference between $m$ and $n$, and consequently that the signal is WSS.
Question: Let $x[n] = v[n] + \frac{1}{2}v[n - 1]$, where $v[n]$ is independently distributed, with mean $\mu = 0$ and variance $\sigma^2 = 1$. Is this signal WSS?
Question (2) and answer

**Question:** Let \( x[n] = v[n] + \frac{1}{2}v[n - 1] \), where \( v[n] \) is independently distributed, with mean \( \mu = 0 \) and variance \( \sigma^2 = 1 \). Is this signal WSS?

**Answer:** The mean of this signal is constant. Now

\[
 r_{xx}[n, m] = \mathcal{E}\left\{ (v[n] + \frac{1}{2}v[n - 1])(v[m] + \frac{1}{2}v[m - 1]) \right\}
\]

\[
 = \mathcal{E}\{v[n]v[m]\} + \frac{1}{2}\mathcal{E}\{v[n - 1]v[m]\} + \frac{1}{2}\mathcal{E}\{v[n]v[m - 1]\} + \frac{1}{4}\mathcal{E}\{v[n - 1]v[m - 1]\}
\]

Apply the independence of the samples, and conclude that the second term equals \( \frac{1}{2} \) for \( n - 1 = m \) and zero otherwise; the third one equals \( \frac{1}{2} \) for \( n = m - 1 \) and zero otherwise; and the sum of the first and of the last one equals \( \frac{5}{4}\mathcal{E}\{v[n]^2\} = \frac{5}{4}\mathcal{E}\{(v[n] - \mu)^2\} = \frac{5}{4}\sigma^2 = \frac{5}{4} \) for \( n = m \), and zero otherwise.

Conclude that \( r_{xx}[n, m] = \frac{1}{2}\delta(n - 1 - m) + \frac{1}{2}\delta(n - m + 1) + \frac{5}{4}\delta(n - m) \), which depends only on \( n - m \). Here \( \delta(k) = 0 \) for \( k \neq 0 \), \( \delta(0) = 1 \). The signal is WSS.

See also [http://en.wikipedia.org/wiki/Dirac_delta_function](http://en.wikipedia.org/wiki/Dirac_delta_function)
The case of ergodic signals

For ergodic signals, the average along the realization equals the average along the ensemble, so

\[
\mu_{x[\cdot]} = \lim_{M \to \infty} \frac{1}{2M + 1} \sum_{n=-M}^{M} x[n]
\]

\[
\sigma_{x[\cdot]}^2 = \lim_{M \to \infty} \frac{1}{2M + 1} \sum_{n=-M}^{M} (x[n] - \mu_{x[\cdot]})^2
\]

\[
r_{xx}[\tau] = \lim_{M \to \infty} \frac{1}{2M + 1} \sum_{n=-M}^{M} x[n]x[n + \tau]
\]

In practice, the limit to infinity cannot be computed, so one considers "large" \(M\)s and computes a finite-time approximation of these quantities.
Properties of the auto-correlation function (1)

Assume for simplicity that $x[\cdot]$ is real. Then:

- If $x[\cdot]$ is stationary, then $\mathcal{E}\{x[n]\} = \mu_x$ and $\mathcal{E}\{(x[n] - \mu_x)(x[n] - \mu_x)\} = \sigma_x^2$ for all $n$; in this case,

$$r_{xx}[0] = \sigma_x^2 + \mu_x^2,$$

since $\mathcal{E}\{x[n]x[n]\} = \mathcal{E}\{x[n]x[n]\} + \mu_x^2 - \mu_x^2 = \mathcal{E}\{(x[n] - \mu_x)(x[n] - \mu_x)\} + \mu_x^2$.

- **Evenness:** $r_{xx}[\tau] = r_{xx}[-\tau]$ ($r_{xx}[\tau] = r_{xx}[-\tau]^*$ for complex signals). This fact has two important consequences:
  - we need to compute $r_{xx}[\cdot]$ only for positive values;
  - the Fourier transform of $r_{xx}[\cdot]$ is real.
Properties of the auto-correlation function (2)

- \( r_{xx}[0] \geq |r_{xx}[\tau]| \) for all \( \tau \): \( r_{xx}[\cdot] \) takes its maximum at \( \tau = 0 \).

The left hand-side graph represents a “typical” autocorrelation; the right hand-side one, the autocorrelation of

\[2\cos(2\pi \omega n + \theta) + 4w[n], \text{ with } w[n] \text{ uniformly distributed between 0 and 1.}\]
Properties of the auto-correlation function (3)

- If a signal is such that any pair of samples $x[m]$ and $x[n]$ are uncorrelated, i.e. $\mathcal{E}\{x[n]x[m]\} = \mathcal{E}\{x[n]\} \mathcal{E}\{x[m]\}$ for $n \neq m$, then it is called a white random process. If the signal is wide sense stationary and $\mathcal{E}\{x[n]\} = 0$ for all $n$, then its autocorrelation function takes the following form:

\[
r_{xx}(\tau)
\]

- Note that the (discrete-time) Fourier transform of $r_{xx}[\cdot]$ in this case is a flat line.
- The terminology “white” is due to the presence of all frequency components with equal strength in a flat spectrum.
- Note that no assumption is made on the distribution of $x[n]$: Gaussian, Poisson, exponential, etc. white noise.
Cross-correlation

The cross-correlation between two samples $x[n]$ and $y[m]$ is defined as:

$$r_{xy}[n, m] := \mathcal{E}\{x[n]y[m]^*\}$$

It can be shown that if the signals are WSS, then the cross-correlation is a function only of the spacing of the samples, and we write

$$r_{xy}[\tau] = \mathcal{E}\{x[n] \cdot y[n - \tau]^*\}$$

Note that the cross-correlation function is in general not commutative:

$$r_{yx}[\tau] = \mathcal{E}\{y[n] \cdot x[n - \tau]^*\} = \mathcal{E}\{y[n'] + \tau \cdot x[n']^*\} = \mathcal{E}\{x[n']^* \cdot y[n' + \tau]\} = r_{xy}[-\tau]^*$$
Cross-correlation techniques in applications

Note that if $y[\cdot] \simeq x[\cdot - m]$, then the cross-correlation between $x[\cdot]$ and $y[\cdot]$ looks like a \textit{m-time instants delayed version of the autocorrelation of $x[\cdot]$} (equivalently, of $y[\cdot]$).

- Delay estimation. Imagine we send a random pulse $x[n]$ and wait for the reflected return signal $y[n]$:

- \textbf{“Matched filtering”:} Compare a received signal with an expected waveform; the cross-correlation will be maximum if the received signal matches up with a desired sequence.
Delay estimation

Two noisy square waves delayed 20 times units from each other. The third graph represents the cross-correlation between the two. Notice that “spike” of the cross-correlation at $t=0$, due to the cross-correlation function of the noise in the two signals; and the peak at $t=20$, corresponding to the separation in time between the two waves.
When analysing time-domain signals, the frequency-domain representation is often very useful. The frequency-domain representation is the Fourier transform of the signal.

Consider a discrete-time signal $x[n]$, the Discrete-Time Fourier Transform (DTFT) $X(e^{j\Omega})$ for $x[n]$ is defined as:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{j\Omega n}$$
Power spectral density

The Fourier transform of the auto-correlation function of a random signal:

\[ R_{xx}(e^{j\Omega}) = \sum_{\tau=-\infty}^{\infty} r_{xx}[\tau] e^{j\Omega \tau} \]

is called the **power spectral density** (PSD). Note that this is a periodic function.

Since PSD and ACF form a Fourier pair, \( r_{xx}[\tau] \leftrightarrow R_{xx}(e^{j\Omega}) \),

\[ r_{xx}[\tau] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_{xx}(e^{j\Omega}) e^{-j\Omega \tau} \, d\Omega \]
Properties of PSDs

- $R_{xx}$ is real.

- It can be shown that $R_{xx}(e^{j\Omega})$ is nonnegative for all $\Omega$.

- The average power of $x[n]$ is (a result similar to Parseval’s)

$$r_{xx}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_{xx}(e^{j\Omega}) \, d\Omega$$

i.e., the scaled area under the PSD.
Examples of PSDs (1)

- PSD for **uncorrelated (“white”)** zero mean noise:
Examples of PSDs (2)

- PSD for **correlated** zero mean noise:

  \[ R_{xx}(e^{j\Omega}) \]

  \[ \sigma_x^2 \]

  area: \( 2\pi \sigma_x^2 \)
Summary

- Auto- and cross-correlation of stationary and ergodic signals;
- WSS signals;
- PSD.

Any questions???